1. **Discrete Mathematics**

**Introduction**

**Discrete Mathematics** is a branch of mathematics involving discrete elements that uses algebra and arithmetic. It is increasingly being applied in the practical fields of mathematics and computer science. It is a very good tool for improving reasoning and problem-solving capabilities. This tutorial explains the fundamental concepts of Sets, Relations and Functions, Mathematical Logic, Group theory, Counting Theory, Probability, Mathematical Induction and Recurrence Relations, Graph Theory, Trees and Boolean Algebra.

Mathematics can be broadly classified into two categories −

* Continuous Mathematics
* Discrete Mathematics

**Continuous Mathematics** is based upon continuous number line or the real numbers. It is characterized by the fact that between any two numbers, there are almost always an infinite set of numbers. For example, a function in continuous mathematics can be plotted in a smooth curve without breaks.

**Discrete Mathematics**, on the other hand, involves distinct values; i.e. between any two points, there are a countable number of points. For example, if we have a finite set of objects, the function can be defined as a list of ordered pairs having these objects, and can be presented as a complete list of those pairs.

**Topics in Discrete Mathematics**

Though there cannot be a definite number of branches of Discrete Mathematics, the following topics are almost always covered in any study regarding this matter −

* Sets, Relations and Functions
* Mathematical Logic
* Group theory
* Counting Theory
* Probability
* Mathematical Induction and Recurrence Relations
* Graph Theory
* Trees
* Boolean Algebra

# **Sets**

German mathematician **G.Cantor** introduced the concept of sets. He had defined a set as a collection of definite and distinguishable objects selected by the means of certain rules or description.

**Set** theory forms the basis of several other fields of study like counting theory, relations, graph theory and finite state machines. In this chapter, we will cover the different aspects of **Set Theory**.

## Set – Definition

A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

### **Some Example of Sets**

* A set of all positive integers
* A set of all the planets in the solar system
* A set of all the states in India
* A set of all the lowercase letters of the alphabet

## Representation of a Set

Sets can be represented in two ways −

* Roster or Tabular Form
* Set Builder Notation

### **Roster or Tabular Form**

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

**Example 1** − Set of vowels in English alphabet, A = {a,e,i,o,u}

**Example 2** − Set of odd numbers less than 10, B = {1,3,5,7,9}

### **Set Builder Notation**

The set is defined by specifying a property that elements of the set have in common. The set is described as A = { x : p(x)}

**Example 1** − The set {a,e,i,o,u} is written as −

A = { x : x is a vowel in English alphabet}

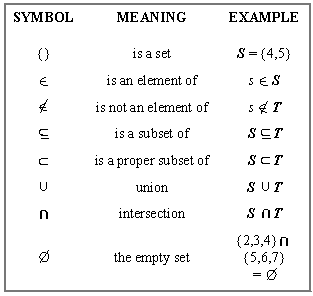
**Example 2** − The set {1,3,5,7,9} is written as −

B = { x : 1≤x<10 and (x%2) ≠ 0}

If an element x is a member of any set S, it is denoted by x ∈ S and if an element y is not a member of set S, it is denoted by y ∉ S.

**Example** − If S = {1, 1.2, 1.7, 2}, 1 ∈ S but 1.5 ∉ S

## Some Common Symbols/Notations



**Some Important Sets**

**N** − the set of all natural numbers = {1, 2, 3, 4, .....}

**Z** − the set of all integers = {....., −3, −2, −1, 0, 1, 2, 3, .....}

**Z+** − the set of all positive integers = {1, 2, 3, 4, .....}

**Q** − the set of all rational numbers

**R** − the set of all real numbers

**W** − the set of all whole numbers

## Numerical Sets

When we define a set, all we have to specify is a common characteristic.

Set of even numbers: {..., -4, -2, 0, 2, 4, ...}  
Set of odd numbers: {..., -3, -1, 1, 3, ...}  
Set of prime numbers: {2, 3, 5, 7, 11, 13, 17, ...}  
Positive multiples of 3 that are less than 10: {3, 6, 9}

## Cardinality of a Set

Cardinality of a set S, denoted by |S|, is the number of elements of the set. If a set has an infinite number of elements, its cardinality is ∞.

**Example** − |{1, 4, 3, 5}| = 4, |{1, 2, 3, 4, 5,…}| = ∞

If there are two sets X and Y,

* |X| = |Y| represents two sets X and Y that have the same cardinality, if there exists a bijective function ‘f’ from X to Y.
* |X| ≤ |Y| represents set X has cardinality less than or equal to the cardinality of Y, if there exists an injective function ‘f’ from X to Y.
* |X| < |Y| represents set X has cardinality less than the cardinality of Y, if there is an injective function f, but no bijective function ‘f’ from X to Y.
* If |X| ≤ |Y| and |X| ≤ |Y| then |X| = |Y|

**Examples:**

* Find the cardinality of set B = {9, 13, 15, 19}

This set contains 4 distinct elements.  Therefore, the cardinal number of set B is 4.  We can also say that set B has a cardinality of 4 or n(B) = 4.

* Find the cardinality of set C = {9, 9, 13, 13, 15, 15, 15, 19}

This set only contains 4 distinct elements.  Therefore, the cardinal number of set C is 4.

* Find the cardinality of set D = {0}

This set contains 1 distinct element.  Therefore, the cardinal number of set D is 1.

* Find the cardinality of set E = {  }

This set contains NO element.  It is the empty set and The cardinal number of set E is 0.

**Cardinality Formula:**

To find the cardinality of a set, we use the following formula which was discovered using the practice of "observation":

http://sites.csn.edu/istewart/Math120/SetTheory/images/imgB11.gif , where

http://sites.csn.edu/istewart/Math120/SetTheory/images/img4.gif is the last number shown in the set

http://sites.csn.edu/istewart/Math120/SetTheory/images/img6.gif is the first number shown in the set

http://sites.csn.edu/istewart/Math120/SetTheory/images/img8.gif is the common difference between all numbers in the set

If we want, we can also memorize the formula in the following form:

http://sites.csn.edu/istewart/Math120/SetTheory/images/img911.gif

* Find the cardinality of set A = {1, 4, 7, ... 49, 52}

The three dots indicate that there are several more numbers in the set that are not listed. These three dots are called an **ellipsis**and indicate that the "missing" elements continue in the same pattern.

In our case, we notice that the difference between the numbers seems to be 3.  You could list all of the elements and then count them, but that might become too cumbersome, especially since the last number is really high.

That is why we will use the formula that allows us to calculate cardinality with ease.

http://sites.csn.edu/istewart/Math120/SetTheory/images/imgB11.gif

In our example, the common difference http://sites.csn.edu/istewart/Math120/SetTheory/images/img8.gif between the numbers is 3, the last number http://sites.csn.edu/istewart/Math120/SetTheory/images/img4.gif is 52, and the first number http://sites.csn.edu/istewart/Math120/SetTheory/images/img6.gif is 1.

Placing these values into the formula gives us

http://sites.csn.edu/istewart/Math120/SetTheory/images/img12.gif

We can easily find that the cardinality http://sites.csn.edu/istewart/Math120/SetTheory/images/img14.gif of the given set is 18.

If we don't believe the outcome of the formula, list all of the elements in the given set and count them.

A = {1, 4, 7, ... 49, 52} = {1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49, 53}

## Table of set symbols/notations with example

|  |  |  |  |
| --- | --- | --- | --- |
| Symbol | Symbol Name | Meaning / definition | Example |
| { } | Set | a collection of elements | A = {3,7,9,14}, B = {9,14,28} |
| | | such that | so that | A = {x | x∈, x<0} |
| A∩B | Intersection | objects that belong to set A and set B | A ∩ B = {9,14} |
| A∪B | Union | objects that belong to set A or set B | A ∪ B = {3,7,9,14,28} |
| A⊆B | Subset | subset has fewer elements or equal to the set | {9,14,28} ⊆ {9,14,28} |
| A⊂B | proper subset / strict subset | subset has fewer elements than the set | {9,14} ⊂ {9,14,28} |
| A⊄B | not subset | left set not a subset of right set | {9,66} ⊄ {9,14,28} |
| A⊇B | Superset | set A has more elements or equal to the set B | {9,14,28} ⊇ {9,14,28} |
| 0  A⊃B | proper superset / strict superset | set A has more elements than set B | {9,14,28} ⊃ {9,14} |
| A⊅B | not superset | set A is not a superset of set B | {9,14,28} ⊅ {9,66} |
| 2A | power set | all subsets of A |  |
|  | power set | all subsets of A |  |
| A=B | Equality | both sets have the same members | A={3,9,14}, B={3,9,14}, A=B |
| Ac | Complement | all the objects that do not belong to set A |  |
| A\B | relative complement | objects that belong to A and not to B | A = {3,9,14}, B = {1,2,3}, A \ B = {9,14} |
| A-B | relative complement | objects that belong to A and not to B | A = {3,9,14}, B = {1,2,3}, A - B = {9,14} |
| A∆B | symmetric difference | objects that belong to A or B but not to their intersection | A = {3,9,14}, B = {1,2,3}, A ∆ B = {1,2,9,14} |
| A⊖B | symmetric difference | objects that belong to A or B but not to their intersection | A = {3,9,14}, B = {1,2,3}, A ⊖ B = {1,2,9,14} |
| *A*∈A | element of | set membership | A={3,9,14}, 3 ∈ A |
| *X*∉A | not element of | no set membership | A={3,9,14}, 1 ∉ A |
| (*a*,*b*) | ordered pair | collection of 2 elements |  |
| A×B | cartesian product | set of all ordered pairs from A and B |  |
| |A| | Cardinality | the number of elements of set A | A={3,9,14}, |A|=3 |
| #A | Cardinality | the number of elements of set A | A={3,9,14}, #A=3 |
| Ø | empty set | Ø = {} | A = Ø |
|  | universal set | set of all possible values |  |
| 0 | natural numbers / whole numbers  set (with zero) | 0 = {0,1,2,3,4,...} | 0 ∈ 0 |
| 1 | natural numbers / whole numbers  set (without zero) | 1 = {1,2,3,4,5,...} | 6 ∈ 1 |
|  | integer numbers set | = {...-3,-2,-1,0,1,2,3,...} | -6 ∈ |
|  | rational numbers set | = {*x*|*x*=*a*/*b*, *a*,*b*∈} | 2/6 ∈ |
|  | real numbers set | = {*x* | -∞ <*x*<∞} | 6.343434 ∈ |
|  | complex numbers set | = {*z*|*z=a*+*bi*, -∞<*a*<∞,      -∞<*b*<∞} | 6+2*i* ∈ |